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Eigenfunctions and asymptotic behavior of Eigenvalues to the given boundary value problem with Eigenparameter in the boundary conditions

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Abstract

In this work, we derived the general solution for nth order linear ordinary differential equations of the form $y^{(n)}(x) + \lambda^n\, y(x) = m(y), x \in [a, b]$for $\lambda \neq 0$, where $m(y) = -p_2(x)y^{(n-2)}(x) - \cdots - p_n(x)y(x)$, and $p_2(x), \dots, p_n(x)$ are continuous functions of x in the interval $[a, b]$, via the method of variation of parameters. Likewise we found the eigenfunctions for the given second boundary value problem [1]-[3] as well as the boundedness of eigenfunctions, and the sign of real part of the eigenvalues have been specified through the sign of one parameter in the boundary conditions. Finally, the asymptotic behavior of eigenvalues to the given problem was studied.

Introduction

The study of boundary value problems containing a spectral parameter in the boundary conditions have many interesting applications, especially in mathematical physics (for example, see **[12]).** Many self-adjoint boundary value problems for a differential equation of the second order for which $\rho(x) = 1$ and including a spectral parameter in the boundary condition were discussed in **[4]**, **[9]**, **[11]**, and **[13]**. Moreover, a singular non-self-adjoint boundary value problem with a discontinuous coefficient and including a spectral parameter in the boundary condition was investigated in **[3]**.

In **[7]** Karwan H. F. Jwamer and Khelan H. Qadr studied asymptotic behaviors of eigenvalues in both cases regular and irregular with estimation of normalized eigenfunctions to the spectral problem:

$$-y'' + q(x)y = \lambda^2\rho(x)y, x \in [0, a], y'(0) = 0, y'(a) + i\lambda y(a) = 0,$$

$$\left(\int_0^a \rho(x)|y(x)|^2 dx\right)^{\frac{1}{2}} = 1.$$

In **[6]** Karwan H. F. and Aryan Ali. M studied the boundedness of the eigenfunctions of the spectral problem of the form:

$-y''(x)+p_1(x)y'(x)+q_1(x)y(x)=\lambda^2\rho(x)y(x),$
with the boundary conditions
$y(0)=0, y'(a)-i\lambda y(a)=0,$

$$\left(\int_0^a \frac{\rho(x)}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}=1,$$

where λ is a spectral parameter, and $\rho(x)$ is a weight function satisfying the Lipschitz condition, $p_1(x)\neq 0, p_1(x)\in C^1[0,a], q_1(x)\in C[0,a]$.

In **[1-2]** and **[5]** the properties of eigenvalues and the estimation of the corresponding eigenfunctions for the boundary value problems consisting the same differential equation
$-y''+q(x)y=\lambda^2\rho(x)y, x\in(0,a)$
but different boundary conditions were studied.

In this paper, we derive the general solution for nth order linear ordinary differential equations of the form $y^{(n)}(x)+\lambda^n\,y(x)=m(y), x\in[a,b]$ for $\lambda\neq 0$, where $m(y)=-p_2(x)y^{(n-2)}(x)-\cdots-p_n(x)y(x)$, and $p_2(x),\dots,p_n(x)$ are continuous functions of x in the interval $[a,b]$, through the method of variation of parameters. Also we find the boundedness of the eigenfunctions of second order boundary value problem [1]-[3] as well as we study the sign of real part of the spectral parameter λ due to the sign of real parameter H in the boundary conditions, furthermore the asymptotic behavior of eigenvalues of the problem:

$$y''(x)+\lambda^2\,y(x)=q(x)y(x)\,, x\in(0,\pi) \tag{1}$$

with the eigenparameter dependent boundary conditions

$$U(y)=y'(0)-hy(0)=0, \tag{2}$$
$$V(y)=y'(\pi)+(\lambda H-H_1)y(\pi)=0, \tag{3}$$

where $q(x)$ is a real valued function, and $h,H,H_1\in\mathbb{R}$, and $\lambda=\delta+i\sigma,\delta,\sigma\in\mathbb{R}$ is a spectral parameter. Assume that the numbering of the roots of $w_k=\sqrt{1}$ is given by
$Re(iw_o\lambda)\le Re(iw_1\lambda)$. Entire complex plane of $\lambda=\delta+i\sigma$ can be divided into four sectors with vertex at $\lambda=0$, so that each sector T_k for different roots of w_k can be ordered so that for

$\lambda\in T_k$ satisfies the inequality $Re(iw_o\lambda)\le Re(iw_1\lambda)$, and the sector T_k in the plane λ is

determined by the inequalities $\frac{(k-1)\pi}{2}\le\arg\lambda\le k\pi, k=1,..,4$.

Derivation of the general solution of $y^{(n)}(x)+\lambda^n\,y(x)=m(y)$ via the method of variation of parameters

The aim of the Derivation depends on the fact that the equation $y^{(n)}(x)+\lambda^n\,y(x)=m(y)$ can be reduced to a certain equivalent integro- differential equation.

Theorem 1

The general solution $y(x)$ of nth order linear ordinary differential equation of the form
$y^{(n)}(x)+\lambda^n\,y(x)=m(y),\qquad x\in[a,b]$
via the method of variation of parameters for $\lambda\neq 0$ can be expressed as

$$y(x)=\sum_{i=1}^{n}c_i\,e^{\lambda w_i x}-\int_a^x\sum_{i=1}^{n}\frac{w_i\,e^{\lambda w_i(x-t)}}{n\,\lambda^{n-1}}\,m(y)dt,$$

where $m(y)=-p_2(x)y^{(n-2)}(x)-\cdots-p_n(x)y(x)$, and $p_2(x),\dots,p_n(x)$ are continuous functions of x in the interval $[a,b]$, and $c_1,c_2,\dots,c_n$ are constants.

Proof

Consider the nth order linear ordinary differential equation of the form

$$L(y) = y^{(n)}(x) + \lambda^n\, y(x) = m(y), x \in [a, b]$$

where $\mathrm{m(y)} = -p_2(x)y^{(n-2)}(x) - \cdots - p_n(x)y(x)$, and $p_2(x), \dots, p_n(x)$ are continuous functions of x in the interval $[a, b]$.
We seek a particular solution of $L(y) = m(y)$ in the form
$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$,
where $\{y_1(x), y_2(x), \dots, y_n(x)\}$ is a known fundamental set of solutions of the complementary equation
$L(y) = y^{(n)}(x) + \lambda^n\, y(x) = 0$,
and $v_1(x), v_2(x), \dots, v_n(x)$ are functions of x to be determined. We begin by imposing the following $n-1$ conditions on $v_1(x), v_2(x), \dots, v_n(x)$:

$$\begin{aligned} &{v'}_1(x)y_1(x) + {v'}_2(x)y_2(x) + \cdots + {v'}_n(x)y_n(x) = 0 \\ &{v'}_1(x)y_1'(x) + {v'}_2(x)y_2'(x) + \cdots + {v'}_n(x)y_n'(x) = 0 \qquad (4) \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &{v'}_1(x)y_1^{(n-2)}(x) + {v'}_2(x)y_2^{(n-2)}(x) + \cdots + {v'}_n(x)y_n^{(n-2)}(x) = 0. \end{aligned}$$

These conditions lead to simple formulas for the first $(n-1)$ derivatives of $y_p(x)$:

$$y_p^{(r)}(x) = v_1(x)y_1^{(r)}(x) + v_2(x)y_2^{(r)}(x) + \cdots + v_n(x)y_n^{(r)}(x),\ 0 \le r \le n-1. \qquad (5)$$

These formulas are easy to remember, since they look as though we obtained them by differentiating equation of $y_p(x)$ $n-1$ times while treating $v_1(x), v_2(x), \dots, v_n(x)$ as constants.
The last equation of equation (5) is

$$y_p^{(n-1)}(x) = v_1(x)y_1^{(n-1)}(x) + v_2(x)y_2^{(n-1)}(x) + \cdots + v_n(x)y_n^{(n-1)}(x).$$

Differentiating this yields

$$y_p^{(n)}(x) = v_1(x)y_1^{(n)}(x) + v_2(x)y_2^{(n)}(x) + \cdots + v_n(x)y_n^{(n)}(x) + v_1'(x)y_1^{(n-1)}(x) + v_2'(x)y_2^{(n-1)}(x) + \dots + v_n'(x)y_n^{(n-1)}(x).$$

Substituting this equation and equation (5) into equation $L(y) = m(y)$ yields

$$v_1(x)\left(y_1^{(n)}(x) + \lambda^n\, y_1(x)\right) + v_2(x)\left(y_2^{(n)}(x) + \lambda^n\, y_2(x)\right) + \cdots + v_n(x)\left(y_n^{(n)}(x) + \lambda^n\, y_n(x)\right) +$$

$$v_1'(x)y_1^{(n-1)}(x) + v_2'(x)y_2^{(n-1)}(x) + \cdots + v_n'(x)y_n^{(n-1)}(x) = m(y).$$

Equations inside the brackets are equals to zero, since $y_1(x), y_2(x), \dots, y_n(x)$ are fundamental solutions of complementary equation $L(y) = 0$. Thus the last equation reduces to

$$v_1'(x)y_1^{(n-1)}(x) + v_2'(x)y_2^{(n-1)}(x) + \cdots + v_n'(x)y_n^{(n-1)}(x) = m(y).$$

Combining this equation with equation (4) show that
$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$
is a solution of equation $L(y) = m(y)$ if

$$\begin{aligned} &{v'}_1(x)y_1(x) + {v'}_2(x)y_2(x) + \cdots + {v'}_n(x)y_n(x) = 0 \\ &{v'}_1(x)y_1'(x) + {v'}_2(x)y_2'(x) + \cdots + {v'}_n(x)y_n'(x) = 0 \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &{v'}_1(x)y_1^{(n-2)}(x) + {v'}_2(x)y_2^{(n-2)}(x) + \cdots + {v'}_n(x)y_n^{(n-2)}(x) = 0 \\ &v_1'(x)y_1^{(n-1)}(x) + v_2'(x)y_2^{(n-1)}(x) + \cdots + v_n'(x)y_n^{(n-1)}(x) = m(y). \end{aligned}$$

These equations can be written in a matrix form as

$$\begin{pmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} v'_1(x) \\ v'_2(x) \\ \vdots \\ v'_n(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ m(y) \end{pmatrix}. \quad (6)$$

The determinant of this system is the Wronskian $W = W(y_1, y_2, \dots, y_n)(x)$ of the fundamental set of solutions $\{y_1(x), y_2(x), \dots, y_n(x)\}$, which has no zeros on $[a, b]$. Solving (6) by Cramer's rule yields

$$v'_j(x) = (-1)^{n-j} \frac{W_j}{W} m(y), (1 \le j \le n)$$

where W_j is the determinant obtained by deleting the last row and jth column of W. Having obtained $v'_1(x), v'_2(x), \dots, v'_n(x)$, we can integrate from a to x to obtain $v_1(x), v_2(x), \dots, v_n(x)$, where $x \in [a, b]$.

The homogeneous linear differential equation $L(y) = 0$ has for $\lambda \ne 0$, the fundamental system $e^{\lambda w_1 x}, e^{\lambda w_2 x}, \dots, e^{\lambda w_n x}$, where $w_1, w_2, \dots, w_n$ represents the first, second,…, nth roots of (-1) respectively. Then the homogeneous solution of complementary equation $L(y) = 0$ is

$y_h(x) = \sum_{i=1}^{n} c_i e^{\lambda w_i x}$, where c_i (for $i = 1,2, \dots, n$) are constants.

We know that the Wronskian of $y_1(x), y_2(x), \dots, y_n(x)$ is defined by

$$W = W(y_1, y_2, \dots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{\lambda w_1 x} & e^{\lambda w_2 x} & \dots & e^{\lambda w_n x} \\ \lambda w_1 e^{\lambda w_1 x} & \lambda w_2 e^{\lambda w_2 x} & \dots & \lambda w_n e^{\lambda w_n x} \\ \vdots & \vdots & & \vdots \\ \lambda^{n-1} w_1^{n-1} e^{\lambda w_1 x} & \lambda^{n-1} w_2^{n-1} e^{\lambda w_2 x} & \dots & \lambda^{n-1} w_n^{n-1} e^{\lambda w_n x} \end{vmatrix}.$$

For every row in the last determinant we factor out the terms $e^{\lambda w_j x}$ $(j = 1, \dots, n)$ and λ^i $(i = 1, \dots, n-1)$ respectively to get

$$W = \lambda^{\frac{n(n-1)}{2}} e^{\lambda (w_1 + w_2 + \dots + w_n) x} \begin{vmatrix} 1 & 1 & \dots & 1 \\ w_1 & w_2 & \dots & w_n \\ \vdots & \vdots & & \vdots \\ w_1^{n-1} & w_2^{n-1} & \dots & w_n^{n-1} \end{vmatrix}.$$

Since $w_1, w_2, \dots, w_n$ are the nth roots of the homogeneous differential equation $y^{(n)}(x) + \lambda^n y(x) = 0$, then by Vieta's formula $w_1 + w_2 + \dots + w_n = 0$, therefore

$$W = \lambda^{\frac{n(n-1)}{2}} \begin{vmatrix} 1 & 1 & \dots & 1 \\ w_1 & w_2 & \dots & w_n \\ \vdots & \vdots & & \vdots \\ w_1^{n-1} & w_2^{n-1} & \dots & w_n^{n-1} \end{vmatrix}.$$

The last determinant is a Vandermonde determinant, hence its value is equal to $\prod_{1 \le i < j \le n} (w_j - w_i)$, so the last equation becomes

$$W = \lambda^{\frac{n(n-1)}{2}} \prod_{1 \le i < j \le n} (w_j - w_i).$$

Now, we use Cramer's rule to find $v_1(x), v_2(x), \dots, v_n(x)$ successively. First, we begin with $v_1(x)$ which is equal to

$$v_1(x) = \int_a^x \frac{W_1}{W} dt = \int_a^x \frac{\begin{vmatrix} 0 & e^{\lambda w_2 t} & \dots & e^{\lambda w_n t} \\ 0 & \lambda w_2 e^{\lambda w_2 t} & \dots & \lambda w_n e^{\lambda w_n t} \\ \vdots & \vdots & & \vdots \\ m(y) & \lambda^{n-1} w_2^{n-1} e^{\lambda w_2 t} & \dots & \lambda^{n-1} w_n^{n-1} e^{\lambda w_n t} \end{vmatrix}}{\lambda^{\frac{n(n-1)}{2}} \prod_{1 \le i < j \le n} (w_j - w_i)} dt.$$

We factor out a term $m(y)$ in the last determinant W_1, and then we expand the resulting $(n-1)\ by\ (n-1)$ determinant in the same way as we have done for expanding the determinant W, after simplification we get

$$v_{\,1}(x) = (-1)^{n+1} \int_a^x \frac{e^{-\lambda w_1 t}}{\lambda^{n-1} \prod_{1<j\le n}(w_j - w_1)}\, m(y)\, dt.$$

Next, we find $v_{\,2}(x)$.

$$v_{\,2}(x) = \int_a^x \frac{W_2}{W}\, dt = \int_a^x \frac{\begin{vmatrix} e^{\lambda w_1 t} & 0 & \dots & e^{\lambda w_n t} \\ \lambda w_1 e^{\lambda w_1 t} & 0 & \dots & \lambda w_n e^{\lambda w_n t} \\ \vdots & \vdots & & \vdots \\ \lambda^{n-1} w_1^{n-1} e^{\lambda w_1 t} & m(y) & \dots & \lambda^{n-1} w_n^{n-1} e^{\lambda w_n t} \end{vmatrix}}{\lambda^{\frac{n(n-1)}{2}} \prod_{1\le i<j\le n}(w_j - w_i)}\, dt.$$

Since

$$W_2 = \begin{vmatrix} e^{\lambda w_1 t} & 0 & \dots & e^{\lambda w_n t} \\ \lambda w_1 e^{\lambda w_1 t} & 0 & \dots & \lambda w_n e^{\lambda w_n t} \\ \vdots & \vdots & & \vdots \\ \lambda^{n-1} w_1^{n-1} e^{\lambda w_1 t} & m(y) & \dots & \lambda^{n-1} w_n^{n-1} e^{\lambda w_n t} \end{vmatrix}$$

$$= (-1)^{n+2} \lambda^{\frac{n(n-1)}{2}-n+1}\, e^{-\lambda w_2 t}\, m(y) \begin{vmatrix} 1 & 1 & \dots & 1 \\ w_1 & w_3 & \dots & w_n \\ \vdots & \vdots & & \vdots \\ w_1^{n-2} & w_3^{n-2} & \dots & w_n^{n-2} \end{vmatrix}.$$

The last determinant is again a Vandermonde determinant, so

$$W_2 = (-1)^{n+2} \lambda^{\frac{n(n-1)}{2}-n+1}\, e^{-\lambda w_2 t}\, m(y) \prod_{1\le i<j\ne 2\le n} (w_j - w_i).$$

By substituting the value of W_2 in the last integral and after simplification, we obtain

$$V_2(x) = (-1)^{n+1} \int_a^x \frac{e^{-\lambda w_2 t}}{\lambda^{n-1} \prod_{1\le j\ne 2\le n}(w_j - w_2)}\, m(y)\, dt.$$

Subsequently, in a similar way we can find $v_{\,3}(x), \dots, v_{\,n}(x)$, then

$$v_{\,n}(x) = (-1)^{n+1} \int_a^x \frac{e^{-\lambda w_2 t}}{\lambda^{n-1} \prod_{1\le i<j\le n-1}(w_j - w_n)}\, m(y)\, dt.$$

One can easily prove that

$$\prod_{1<j\le n} (w_j - w_1) = (-1)^n \frac{n}{w_1},$$

and generally

$$\prod_{1\le j\le n-1}(w_j-w_n)=(-1)^n\,\frac{n}{w_n}.$$

Then the equations $v_1(x), v_2(x), \dots, v_n(x)$ reduces to

$$v_{\,1}(x)=-\int_a^x \frac{w_1\,e^{-\lambda\,w_1\,t}}{n\,\lambda^{n-1}}\,m(y)\,dt,$$

$$v_{\,2}(x)=-\int_a^x \frac{w_2\,e^{-\lambda\,w_2\,t}}{n\,\lambda^{n-1}}\,m(y)\,dt,$$

:

In general

$$v_{\,n}(x)=-\int_a^x \frac{w_n\,e^{-\lambda\,w_n\,t}}{n\,\lambda^{n-1}}\,m(y)\,dt.$$

By putting the values of $y_i(x)\,v_i(x)\;for\;i=1,2,\dots,n$ in equation $y_p(x)=\sum_{i=1}^{n} y_i(x)\,v_i(x)$, we deduce the particular solution of a differential equation $L(y)=m(y)$

$$y_p(x)=-\int_a^x \sum_{i=1}^{n}\frac{w_i\,e^{\lambda\,w_i(x-t)}}{n\,\lambda^{n-1}}\,m(y)dt.$$

Hence, the general solution of the given differential equation $L(y)=m(y)$ is

$$y(x)=y_h(x)+y_p(x)$$

$$y(x)=\sum_{i=1}^{n} c_i\,e^{\lambda\,w_i x}-\int_a^x \sum_{i=1}^{n}\frac{w_i\,e^{\lambda\,w_i\,(x-t)}}{n\,\lambda^{n-1}}\,m(y)dt.$$

Thus the proof of theorem 1 is finished.

Example

Consider the second order linear ordinary differential equation
$L(y)=y''(x)+\lambda^2 y(x)=e^x=m(y), x\in(0,\pi)$
with the boundary conditions
$U(y)=y'(0)-hy(0)=0,$
$V(y)=y'(\pi)+(\lambda H-H_1)y(\pi)=0,$
then we find the general solution for the given differential equation $L(y)=e^x$ by using the form of $y(x)$ which we have obtained in Theorem 1

Solution

The homogeneous solution of the complementary equation $y''(x)+\lambda^2\,y(x)=0$ is
$y_h(x)=c_1e^{i\lambda\,x}+c_2e^{-i\lambda\,x},$
where $w_1=i$, $w_2=-i$, and c_1, c_2 are constants.
Next, we try to find a particular solution $y_p(x)$ for the differential equation $L(y)=e^x$.

$$y_p(x)=-\int_a^x \frac{w_1e^{\lambda w_1(x-t)}}{2\lambda}\,m(y)dt-\int_a^x \frac{w_2e^{\lambda w_2(x-t)}}{2\lambda}\,m(y)\,dt$$

$$= -\int_0^x \frac{i\,e^{i\lambda(x-t)}}{2\lambda}\,e^t\,dt - \int_0^x \frac{-i\,e^{-i\lambda(x-t)}}{2\lambda}\,e^t\,dt = -\frac{ie^{i\lambda x}}{2\lambda}\int_0^x e^{(1-i\lambda)t}\,dt + \frac{ie^{-i\lambda x}}{2\lambda}\int_0^x e^{(1+i\lambda)t}\,dt$$

$$= -\frac{i\,e^{i\lambda x}}{2\lambda\,(1-i\lambda)}\,[e^{(1-i\lambda)t}]_0^x + \frac{i\,e^{-i\lambda x}}{2\lambda\,(1+i\lambda)}\,[e^{(1+i\lambda)t}]_0^x$$

$$y_p(x) = \frac{e^x}{(1+\lambda^2)} + i\,\frac{(1+i\lambda)e^{i\lambda x}}{2\lambda(1+\lambda^2)} - i\,\frac{(1-i\lambda)e^{-i\lambda x}}{2\lambda(1+\lambda^2)}.$$

Then the general solution of the given differential equation $L(y) = e^x$ is

$$y(x) = c_1e^{i\lambda\,x} + c_2e^{-i\lambda\,x} + \frac{e^x}{(1+\lambda^2)} + i\,\frac{(1+i\lambda)e^{i\lambda x}}{2\lambda(1+\lambda^2)} - i\,\frac{(1-i\lambda)e^{-i\lambda x}}{2\lambda(1+\lambda^2)}.$$

Here in our example, we take $h = 1, H = 1$, and $H_1 = 1$, so the given boundary conditions become:
$U(y) = y'(0) - y(0) = 0,$
$V(y) = y'(\pi) + (\lambda - 1)y(\pi) = 0.$
Now, we apply these boundary conditions on the general solution to find c_1 and c_2.

$$y'(x) = c_1i\lambda e^{i\lambda\,x} - c_2i\lambda e^{-i\lambda\,x} + \frac{e^x}{(1+\lambda^2)} - \frac{(1+i\lambda)e^{i\lambda x}}{2(1+\lambda^2)} - \frac{(1-i\lambda)e^{-i\lambda x}}{2(1+\lambda^2)}.$$

From boundary condition $y'(0) - y(0) = 0$, we get

$$(i\lambda - 1)c_1 - (i\lambda + 1)c_2 = -\frac{1}{1+\lambda^2}\,. \tag{7}$$

And from applying boundary condition $y'(\pi) + (\lambda - 1)y(\pi) = 0$, we obtain

$$2\lambda(1+\lambda^2)\big((\lambda-1)+i\lambda\big)e^{i2\lambda\pi}c_1 + 2\lambda\,(1+\lambda^2)\big((\lambda-1)-i\lambda\big)c_2$$
$$= \big((i+1)\lambda - i\big)(1-i\lambda) - \big((i-1)\lambda - i\big)(1+i\lambda)e^{i2\lambda\pi} - 2\lambda^2e^{(1+i\lambda)\pi}. \tag{8}$$

From equations (7) and (8) we conclude that

$$c_1 = \frac{-1+\lambda(1-i)}{(\lambda^2+1)(\lambda + e^{i2\lambda\pi} - \lambda e^{i2\lambda\pi} + \lambda^2(-i-1) - 1 + \lambda^2(1-i)\,e^{i2\lambda\pi}} -$$
$$\frac{2\lambda^2\,e^{(1+i\lambda)\pi} + (-1+\lambda i)(\lambda(i+1)-i) - (1+\lambda i)(\lambda(1-i)+i)e^{i2\lambda\pi}}{2\lambda(\lambda+i)(i\lambda - ie^{i2\lambda\pi} - i\lambda e^{i2\lambda\pi} + \lambda^2(i+1)e^{i2\lambda\pi} + (1-i)\lambda^2 - i\,)}\,, \tag{9}$$

$$c_2 = -\frac{(-1+(i+1)\lambda)\,e^{i2\lambda\pi}}{(\lambda^2+1)(\lambda + e^{i2\lambda\pi} - \lambda e^{i2\lambda\pi} + (-i-1)\lambda^2 - 1 + (1-i)\lambda^2e^{i2\lambda\pi})} +$$
$$\frac{\big(2\lambda^2\,e^{(1+i\lambda)\pi} + (-1+\lambda i)(\lambda(i+1)-i) - (1+\lambda i)(\lambda(1-i)+i)i\,e^{i2\lambda\pi}\big)}{2\lambda(\lambda-i)(\lambda + e^{i2\lambda\pi} - \lambda e^{i2\lambda\pi} + \lambda^2(-i-1) - 1 + \lambda^2(1-i)\,e^{i2\lambda\pi})}\,. \tag{10}$$

Hence

$$y(x) = c_1e^{i\lambda\,x} + c_2e^{-i\lambda\,x} + \frac{e^x}{(1+\lambda^2)} + i\,\frac{(1+i\lambda)e^{i\lambda x}}{2\lambda(1+\lambda^2)} - i\,\frac{(1-i\lambda)e^{-i\lambda x}}{2\lambda(1+\lambda^2)}\,,$$

c_1 and c_2 were defined from equations (9) and (10).

Note

If we have a second order linear differential equation of the form
$y''(x) + \lambda^2y(x) = m(y),$
where $m(y) = q(x)y(x)$, so in this case the equation of the general solution

$$y(x) = \sum_{i=1}^{n} c_i\, e^{\lambda\, w_i x} - \int_a^x \sum_{i=1}^{n} \frac{w_i\, e^{\lambda\, w_i\,(x-t)}}{n\, \lambda^{n-1}}\, m(y) dt,$$

will be Volterra integral equations of the second kind and can be solved by the methods of solution of Volterra integral equations of the second kind like resolvent kernel –Neumann series method.

Lemma 1

If $y(x)$ is an eigenfunction of the differential equation (1) corresponding to the eigenvalue λ, then $|y(x)| \le 1$, if $|\lambda| \to \infty$.

Proof

Let us consider the second order linear differential equation

$$y''(x) + \lambda^2\, y(x) = q(x)y(x)\,.$$

In **[8]** NAIMARK proved that the second order linear differential equation (1) has two linear independent solutions $y_k(x,\lambda)$ $(for\ k = 1{,}2)$, which can be expressed as

$$y_k(x,\lambda) = e^{(\mp i)\,\lambda x}\left(1 + O\left(\frac{1}{\lambda}\right)\right), \text{then}$$

$$|y_k(x,\lambda)| = \left|e^{(\mp i)\,\lambda x}\left(1 + O\left(\frac{1}{\lambda}\right)\right)\right| = \left|e^{(\mp i)\,\lambda x}\right|\left|1 + O\left(\frac{1}{\lambda}\right)\right|$$

$$\le\ |1| + \left|O\left(\tfrac{1}{\lambda}\right)\right|,$$

or

$$|y_k(x,\lambda)| - |1| \le\ \left|O\left(\frac{1}{\lambda}\right)\right|.$$

And from above definition of (order big O) **[10]**, we can say that ∃ a constant $k \ge 0,$ and

$\delta > 0\ \ni$ for $0 < |\lambda - \lambda_o| <\ \delta,$ then $|y_k(x,\lambda)| - |1| \le k\left|\frac{1}{\lambda}\right|.$

Or

$$|y_k(x,\lambda)| \le |1| + k\left|\frac{1}{\lambda}\right| = |1| + \frac{k}{|\lambda|}.$$

So, if $|\lambda| \to\ \infty,\ then\ \frac{k}{|\lambda|} \to 0,$ hence

$$|y_k(x,\lambda)| \le 1.$$

Thus the proof of Lemma 1 is ended.

Theorem 2

Let $\lambda = \delta + i\sigma$ be an eigenvalue of the given problem (1)-(3) corresponding to the eigenfunction $y(x)$, if $\sigma \ne 0$, then the real part of λ is:

(1) Non-negative if $H \ge 0$. (2) Negative if $H < 0$.

Proof

The given problem with the boundary conditions will become:

$$-y''(x)+q(x)y(x)=\lambda^2 y(x), \qquad x\in(0,\pi) \tag{1}$$

$$U(y)=y'(0)-hy(0)=0, \tag{2}$$

$$V(y)=y'(\pi)+(\lambda H-H_1)y(\pi)=0. \tag{3}$$

Taking the complex conjugates of all terms in (1), (2) and (3), we obtain

$$-\bar{y}''(x)+q(x)\,\bar{y}(x)=\bar{\lambda}^2\,\bar{y}(x), \qquad x\in(0,\pi) \tag{11}$$

$$\bar{y}'(0)-h\bar{y}(0)=0, \tag{12}$$

$$\bar{y}'(\pi)+\left(\bar{\lambda}\,H-H_1\right)\bar{y}(\pi)=0. \tag{13}$$

The above equations (11), (12) and (13) show that $\bar{y}(x)$ is the eigenfunction corresponding to the eigenvalue $\bar{\lambda}=\delta-i\sigma$.

Multiplying equation (1) by $\bar{y}(x)$ and equation (11) by $y(x)$ and subtracting, we obtain

$$\left(\lambda^2-\bar{\lambda}^2\right)|y(x)|^2=\ \bar{y}''(x)y(x)-y''(x)\bar{y}(x)$$

Or

$$\left(\lambda-\bar{\lambda}\right)\left(\lambda+\bar{\lambda}\right)|y(x)|^2=\frac{d}{dx}\left(\bar{y}'(x)y(x)-y'(x)\bar{y}(x)\right).$$

Integrating both sides of the above equation with respect to x from 0 to π, we thus get

$$\left(\lambda-\bar{\lambda}\right)\left(\lambda+\bar{\lambda}\right)\int_0^\pi |y(x)|^2\ dx=[\bar{y}'(x)y(x)-y'(x)\bar{y}(x)]_0^\pi$$

$$\left(\lambda-\bar{\lambda}\right)\left(\lambda+\bar{\lambda}\right)\int_0^\pi |y(x)|^2\ dx=[\bar{y}'(\pi)\,y(\pi)-y'(\pi)\bar{y}(\pi)-\left(\bar{y}'(0)\,y(0)-y'(0)\bar{y}(0)\right). \tag{14}$$

Multiplying equation (2) by $\bar{y}(0)$ and equation (12) by $y(0)$ and then subtracting, we acquire

$$\bar{y}'(0)\,y(0)-y'(0)\bar{y}(0)=0. \tag{15}$$

Again, multiplying equation (3) by $\bar{y}(\pi)$ and equation (13) by $y(\pi)$ and then subtracting, yields

$$\bar{y}'(\pi)\,y(\pi)-y'(\pi)\bar{y}(\pi)=\left(\lambda-\bar{\lambda}\right)H\,|y(\pi)|^2. \tag{16}$$

By setting equations (15), (16) in equation (14), we deduce

$$\left(\lambda-\bar{\lambda}\right)\left(\lambda+\bar{\lambda}\right)\int_0^\pi |y(x)|^2\ dx=\left(\lambda-\bar{\lambda}\right)H\,|y(\pi)|^2.$$

Since, $\lambda-\bar{\lambda}=i2\sigma$ and $\lambda+\bar{\lambda}=2\delta,$ hence

$2\delta\sigma \int_0^\pi |y(x)|^2 \ dx = \sigma\, H|y(\pi)|^2$, this implies that

$\delta = \frac{1}{2} H|y(\pi)|^2 \int_0^\pi |y(x)|^2 \ dx$, and since $|y(\pi)|^2, \int_0^\pi |y(x)|^2 \ dx > 0$, thus

(1) If $H \geq 0$, then the real part of $\lambda = \delta \geq 0$, and

(2) If $H < 0$, then the real part of $\lambda = \delta < 0$.

Hence, the proof of theorem 2 is completed.

Theorem 3

The asymptotic behavior of eigenvalues of the given problem (1)-(3) in the sector T_o has the form $\lambda_m = \left(m - \frac{i}{2\pi} Lnf(\lambda) + O\left(\frac{1}{m}\right)\right)$, and in the sector T_1 asymptotic behavior of the

spectrum has the form $\lambda_m = \left(m + \frac{i}{2\pi} Lnf(\lambda) + O\left(\frac{1}{m}\right)\right)$, where $\mathrm{m} = \mathrm{N}, \mathrm{N}+1, \dots, \mathrm{N}$ is a

Natural number and $f(\lambda) = \frac{(h-i\lambda)^2 (H_2 - i\lambda)^2}{(h^2+\lambda^2)(H_2^2+\lambda^2)}$, where $(\mathrm{h}^2 + \lambda^2)(H_2^2 + \lambda^2) \neq 0,\ H_2 = H - H_1$.

Proof

We know the given problem with boundary conditions is defined as:

$-y_k''(x) + q(x) y_k(x) = \lambda^2 y_k(x), \qquad x \in (0, \pi)$

$U(y_k) = y_k'(0) - h y_k(0) = 0,$

$V(y_k) = y_k'(\pi) + (\lambda H - H_1) y_k(\pi) = 0$, for $k = 0,1$.

Since $y_k(x) = y_k(x,\lambda)$are linearly independent solutions of the given second order linear ordinary spectral problem, so they are satisfied the above problem.

In **[8]** NAIMARK proved that $y_k(x,\lambda)$ has the form

$y_k(x,\lambda) = e^{(\mp i)\,\lambda x}\left(1 + O\left(\frac{1}{\lambda}\right)\right), for\ k = 0,1.$

Let us consider the determinant of $\Delta(\lambda)$, which is defined by

$\Delta(\lambda) = \left|U\left(y_j\right)\right|_{k,j=0,1}.$

The following results can be obtained by using the equation of $y_k(x,\lambda)$ and the above boundary conditions.

$U(y_o) = (i\lambda - h)\left(1 + O\left(\frac{1}{\lambda}\right)\right),$

$U(y_1) = (-i\lambda - h)\left(1 + O\left(\frac{1}{\lambda}\right)\right)$,

$V(y_o) = \big((H + i)\lambda - H_1\big)e^{i\lambda\pi}\left(1 + O\left(\frac{1}{\lambda}\right)\right)$,

$V(y_1) = \big((H - i)\lambda - H_1\big)e^{-i\lambda\pi}\left(1 + O\left(\frac{1}{\lambda}\right)\right)$.

If we put $[1] = \left(1 + O\left(\frac{1}{\lambda}\right)\right)$, then the determinant of $\Delta(\lambda)$ becomes

$$\Delta(\lambda) = \begin{vmatrix} (i\lambda - h)[1] & (-i\lambda - h)[1] \\ \big((H + i)\lambda - H_1\big)e^{i\lambda\pi}[1] & \big((H - i)\lambda - H_1\big)e^{-i\lambda\pi}[1] \end{vmatrix}.$$

The eigenvalues λ are the zeros of the function $\Delta(\lambda) = 0$, hence the determinant of $\Delta(\lambda)$ reduced to

$$(i\lambda - h)\big((H - i)\lambda - H_1\big)e^{-i\lambda\pi}[1] - (-i\lambda - h)\big((H + i)\lambda - H_1\big)e^{i\lambda\pi}[1] = 0,$$

or

$$(h + i\lambda)\big((H - H_1) + i\lambda\big)e^{i\,2\lambda\pi} = (h - i\lambda)\big((H - H_1) - i\lambda\big).$$

If we put $H_2 = H - H_1$, therefore, the last equation turn into

$$e^{i\,2\lambda\pi} = \frac{(h - i\lambda)(H_2 - i\lambda)}{(h + i\lambda)(H_2 + i\lambda)} = \frac{(h - i\lambda)^2(H_2 - i\lambda)^2}{(h^2 + \lambda^2)(H_2^2 + \lambda^2)}.$$

Suppose $f(\lambda) = \frac{(h - i\lambda)^2(H_2 - i\lambda)^2}{(h^2 + \lambda^2)(H_2^2 + \lambda^2)}$, hence

$$e^{i\,2\lambda\pi} = f(\lambda) \ \rightarrow\ i2\lambda\pi = Ln\, f(\lambda) + 2m\pi i + O\left(\frac{1}{m}\right),$$

$$\lambda_m = \frac{1}{i2\pi}\left(Ln\, f(\lambda) + 2m\pi i + O\left(\frac{1}{m}\right)\right),$$

or

$\lambda_m = \left(m - \frac{i}{2\pi}\, Ln\, f(\lambda) + O\left(\frac{1}{m}\right)\right)$, where $m = N, N + 1, \ldots$ (N is anatural number).

Thus, the asymptotic behavior of spectrum in the sector T_o has the form

$\lambda_m = \left(m - \frac{i}{2\pi}\, Ln\, f(\lambda) + O\left(\frac{1}{m}\right)\right)$, where $m = N, N + 1, \ldots$ (N is anatural number),

and in the sector T_1

$\lambda_m = \left(m + \frac{i}{2\pi} Lnf(\lambda) + O\left(\frac{1}{m}\right)\right)$, where $m = N, N + 1, \ldots$ (N is anatural number).

Conclusion

In the present work, we derived the general solution for nth order linear ordinary differential equations of the form $y^{(n)}(x) + \lambda^n y(x) = m(y)$ for $\lambda \neq 0$, by means of the method of variation of parameters, moreover we studied the boundedness of eigenfunctions of the given boundary problem (1)-(3) , and we proved that the sign of real part of the eigenvalues changed by means of the sign of one parameter in the boundary conditions. Finally, we showed the asymptotic behavior of eigenvalues to the given problem.

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